In–out intermittency in PDE and ODE models of axisymmetric mean–field dynamos

Eurico Covas[1], Reza Tavakol[1], Peter Ashwin[2], Andrew Tworkowski[3] and John M. Brooke[4]

1.Astronomy Unit, Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS, UK
2.Department of Mathematical and Computing Sciences, University of Surrey, Guildford GU2 5XH, UK
3.Mathematics Research Centre, Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS, UK
4.HPC Support Group Manchester, Computing Centre, Oxford Road, Manchester M13 9PL, UK

(February 5, 2008)

Employing some recent results in dynamics of systems with invariant subspaces we find evidence in both truncated and full axisymmetric mean–field dynamo models of a recently discovered type of intermittency, referred to as in–out intermittency. This is a generalised form of on–off intermittency that can occur in systems that are not skew products. As far as we are aware this is the first time detailed evidence has been produced for the occurrence of a particular form of intermittency for such deterministic PDE models and their truncations. The specific signatures of this form of intermittency make it possible in principle to look for such behaviour in solar and stellar observations. Also in view of its generality, this type of intermittency is likely to occur in other physical models with invariant subspaces.

An important feature of the Sun and stars is their variability on a wide range of time scales [1]. In addition to the directly observed nearly periodic magnetic solar cycles, with an average period of 22 years, a particularly important feature of this variability is the presence of episodes, such as the so called *Maunder Minimum* of the 17th century, during which solar activity (as deduced from sunspot numbers) virtually vanished [2]. The proxy data seems to indicate that this episode was not singular, but was preceded by numerous similar events occurring intermittently with an intermediate time scale of the order of 10^2 years. There is also some evidence suggesting similar variability in solar type stars [3].

Given the absence of naturally occurring mechanisms with such time scales in the Sun and stars, as well as evidence for the presence of non-linear phenomena in stellar and solar magnetic activity, one of the more plausible suggestions has been to associate this type of variability with some form of dynamical *intermittency* in the magnetohydrodynamical dynamos operating in the solar/stellar interiors [4–7].

The complexity of the nonlinear partial differential equations (PDE) modelling these regimes has essentially led to three approaches, in turn employing:
1. Low dimensional models which encode the main features of dynamo models and use, for example, normal form theory (see e.g. [8]). These models, however, may not necessarily possess properties which we shall argue below are generic in axisymmetric dynamos.

2. Low dimensional truncations [7,9–11].
3. Direct numerical integration of the full PDE models (e.g. [12,13]).

Models of type 1 have produced many useful insights into the nature of dynamo models. Models of type 2 have been shown conclusively to be capable of producing a number of different types of intermittency, including crisis intermittency [14] and type I intermittency [15]. Numerical studies of models of type 3 have produced a rich set of dynamical behaviours (see e.g. [13,16]), as well as intermittent types of behaviour [17–19]. The problem has, however, been how to make precise the nature of these numerically produced dynamical modes of behaviour. This is crucial for two reasons: firstly to be sure that these models are indeed capable of producing intermittent behaviour and secondly to use this information to characterise precisely their nature in order to compare their predicted signatures with observational data.

The main difficulty has been the absence of an appropriate theoretical framework underlying such systems. Here, we shall use recent results in the transverse stability of attractors with invariant subspaces [15,20] to demonstrate the presence in axisymmetric mean–field dynamo models of a recently discovered type of intermittency referred to as *in–out intermittency*.

Most studies of stellar dynamos have relied on mean–field theory, which is the approach we adopt here. The standard mean–field dynamo equation is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{u} \times \mathbf{B} + \alpha \mathbf{B} - \eta_t \nabla \times \mathbf{B} \right), \qquad (1)$$

where $\mathbf{B}$ and $\mathbf{u}$ are the mean magnetic field and mean velocity respectively and the turbulent magnetic diffusivity η_t and the coefficient α arise from the correlation of small scale turbulent velocities and magnetic fields [21]. In axisymmetric geometry, eq. (1) is solved by splitting the magnetic field into meridional and azimuthal components, $\mathbf{B} = \mathbf{B_p} + \mathbf{B}_\phi$, and expressing these components in terms of scalar field functions $\mathbf{B_p} = \nabla \times A\hat{\phi}$, $\mathbf{B}_\phi = B\hat{\phi}$.

In the following we shall use a finite order truncation of the one dimensional version of equation (1) along with a time dependent form of α, obtained by using a spectral expansion, of the form:

$$\frac{dA_n}{dt} = -n^2 A_n + \frac{D}{2}(B_{n-1} + B_{n+1}) +$$

1

$$\sum_{m=1}^{N}\sum_{l=1}^{N}\mathcal{F}(n,m,l)B_m C_l,$$

$$\frac{dB_n}{dt} = -n^2 B_n + \sum_{m=1}^{N}\mathcal{G}(n,m)A_m,$$

$$\frac{dC_n}{dt} = -\nu n^2 C_n - \sum_{m=1}^{N}\sum_{l=1}^{N}\mathcal{H}(n,m,l)A_m B_l. \qquad (2)$$

where A_n, B_n and C_n derive from the spectral expansion of the magnetic field $\mathbf{B}$ and α respectively, $\mathcal{F}, \mathcal{H}$ and $\mathcal{G}$ are coefficients expressible in terms of m, n and l, N is the truncation order, D is the dynamo number and ν is the Prandtl number (see [15] for details).

We first of all briefly discuss the main features necessary for the appearance of in–out intermittency [15,20] and introduce the necessary vocabulary in the context of axisymmetric dynamo models.

I. Invariant subspaces: In addition to simplifying the resulting equations and hence aiding the numerical integration, the presence of symmetry forces dynamo models to possess invariant subspaces. For example, the truncated model (2) with $N = 4$ is a 12–dimensional system of ordinary differential equations (ODE) with two 6–dimensional symmetric and antisymmetric invariant subspaces given by $M_S = \{0, B_1, 0, A_2, 0, C_2, 0, B_3, 0, A_4, 0, C_4\}$ and $M_A = \{A_1, 0, 0, 0, B_2, C_2, A_3, 0, 0, 0, B_4, C_4\}$ respectively. Similarly the PDE model (1) possesses invariant subspaces $M_S = \{B(\theta) = B(-\theta), A(\theta) = -A(-\theta)\}$ and $M_A = \{B(\theta) = -B(-\theta), A(\theta) = A(-\theta)\}$, where θ is the latitude. As a result, a trajectory starting in either subspace remains in that subspace for all times.

II. Non–skew product: If the dynamics can be written as an evolution in an invariant subspace forcing the transverse dynamics (skew product) we cannot have proper in–out intermittency but can have on–off intermittency. It is a generic property that we do not have a skew product.

A related feature that seems to aid the appearance of in–out intermittency is the presence of *non–normal* parameters that vary the system within the invariant subspace as well as outside it. In the case of the truncated system (2), both the dynamo number D and the Prandtl number ν are clearly non–normal parameters, as they enter the equations for A_n and C_n respectively and these variables in turn are a part of the invariant subspaces M_S and M_A. We note that generic parameters are non–normal.

Recent studies of systems with these features [15,20] show the presence of a number of novel types of dynamical behaviour, including in–out intermittency.

To characterise in–out intermittency, it is best to contrast it with on–off intermittency [22] as both types of intermittency occur in systems with invariant subspaces. On-off intermittency can occur as the result of an instability of an attractor in an invariant subspace. It manifests itself as an attractor whose trajectories get arbitrar-

ily close to an attractor for the system in the invariant subspace while making occasional large deviations away from it intermittently [22]. It can be modelled by a biased random walk of logarithmic distance from the invariant subspace [22].

We say an attractor A exhibits in–out intermittency to the invariant subspace M_A (or M_S), if the following are true [20]:

1. The intersection $A_0 = A \cap M_A$ (or $A_0 = A \cap M_S$) is not necessarily a minimal attractor, i.e. there can be proper subsets of A_0 that are attractors (for on–off intermittency A_0 is assumed to be minimal). This means that there can be different invariant sets in A_0 associated with attraction and repulsion transverse to A_0 (hence the name in–out) and these growing and decaying phases come about through different mechanisms within M_A (or M_S).
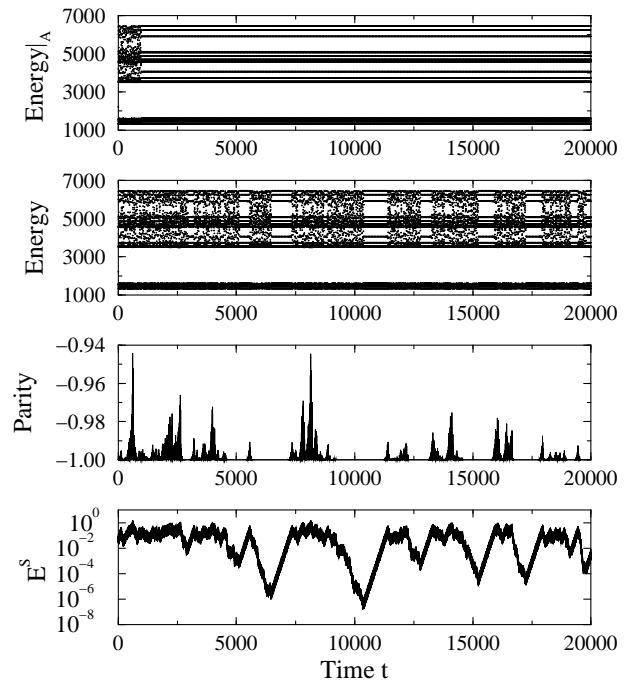

FIG. 1. In–out intermittency in the ODE model (2) with $N = 4$ and parameter values $D = 177.7$ and $\nu = 0.47$. The energy and parity are given by $E = E^A + E^S$ and $P = (E^S - E^A)/E$ respectively, where E^A and E^S are the antisymmetric and symmetric parts of the magnetic field energy with respect to the rotational equator ("antisymmetric" (P=-1) and "symmetric" (P=+1)). The top panel shows evolution of an initial condition in M_A and the other panels a nearby initial condition not in M_A. In these panels, we have taken a Poincaré section at $A_1 = 0$, reducing the ODE to a map for clarity and comparison.

2. The minimal attractors in the invariant subspace are not necessarily chaotic; they can be periodic orbits or equilibria. Furthermore, the trajectory remains close to this attractor during the moving away or "out" phases, with the important consequence that during these out phases the trajectory can shadow for example a periodic

orbit while drifting away at an exponential rate [20] (see also [23]).

3. The asymptotic scaling of the distribution of laminar phases in the in–out case can have two contributions:

$$P_n \sim \alpha n^{-3/2} e^{(-\beta n)} + \gamma e^{(-\delta n)} = I_1 + I_2, \qquad (3)$$

where α, β, γ and δ are positive real constants depending on the bias of the random walk modelling the "in" chain and the probability of leaking into the deterministic "out" chain [20]. The term I_1 is the one from biased on–off intermittency. The extra term I_2 can cause an identifiable shoulder to develop at large n which can help to statistically distinguish this type of intermittency from on–off intermittency.

4. If the system has a skew–product structure, in–out intermittency reduces to on–off intermittency [20].

Because of the above considerations, we expect that in–out intermittency will be more generally visible in such dynamo models. Here we look at mean–field dynamo models and find in–out intermittency in such systems.

Since the ODE models are more transparent, we first look at the truncated system (2) with $N = 4$. Fig. 1 shows an example of in–out intermittency in this system at parameter values $D = 177.7$ and $\nu = 0.47$. One can see clearly the "periodic" out phases (second panel), where the trajectory of the full system shadows the periodic orbit in the antisymmetric invariant subspace M_A (top panel). Also we can clearly see the exponential growth of the amplitudes of the transverse variables through several orders of magnitude (lower panel). Furthermore, the calculated scaling of the distribution of the laminar phases, shown in Fig. 2, is compatible with a curve obtained by fitting the parameters in the scaling law (3).
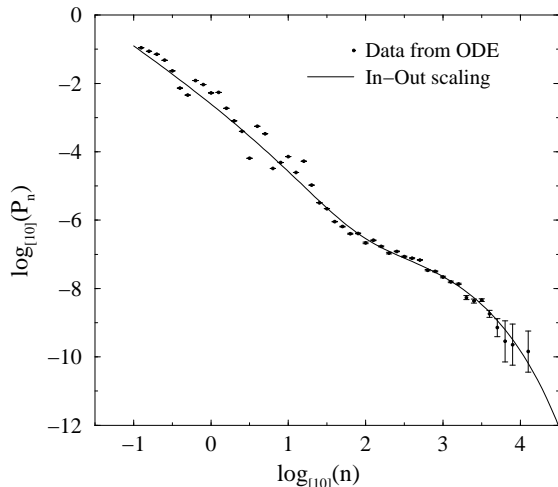

FIG. 2. Scaling of laminar phases for the model (2) with $N = 4$, $D = 177.7$ and $\nu = 0.47$, where the shoulder at large laminar phases (which is the influence of I_2 and a characteristic of in–out) may be discerned.

These signatures, namely the periodicity of the attractor of the system restricted to the invariant submani-

fold, the periodic locking and the exponential growth of the out phases and the compatibility with the scaling (3) clearly show the occurrence of in–out intermittency for the truncated dynamo systems. To show that this also happens in the full PDE axisymmetric mean–field dynamo models, we first of all recall that such systems possess the ingredients required for the presence of in–out intermittency, namely the existence of invariant submanifolds (symmetric or antisymmetric), non–skew product structure and non–normal parameters.

Guided by recent results [17], we integrated the mean–field dynamo equations using the code described in [13] and implemented by [16].
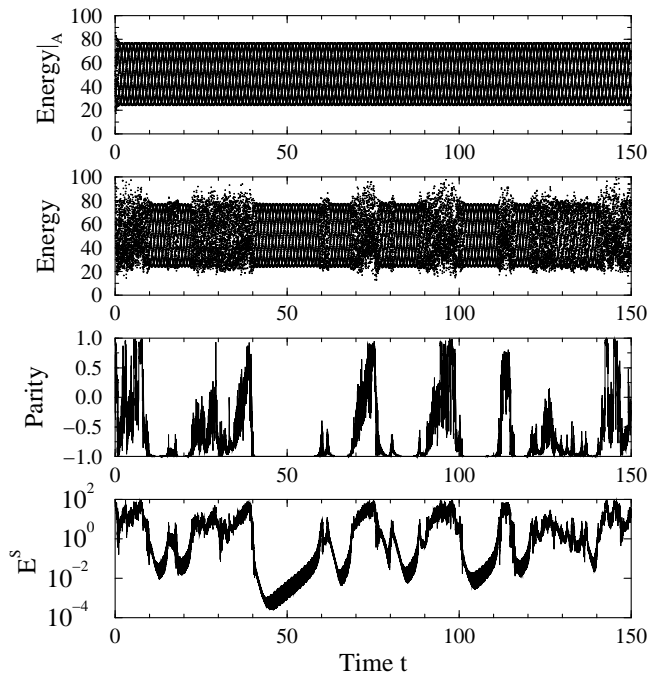

FIG. 3. In–out intermittency in the axisymmetric PDE mean–field dynamo model (1). The parameters used were $r_0 = 0.4$, $C_\alpha = 1.942$, $C_\Omega = -10^5$ ($C_\alpha C_\Omega \sim D$), $f = 0.0$, with the usual algebraic form of $\alpha = \alpha_0/(1+\mathbf{B}^2)$ (see [17] for details of the parameters). To enhance visually the periodic locking we time sample the series in the two upper panels.

Fig. 3 and Fig. 4 give examples of in–out intermittency in these PDE models with different algebraic forms of α. As can be seen this behaviour can occur with the invariant submanifold being either antisymmetric (Fig. 3) or symmetric (Fig. 4). We have also obtained similar results using a dynamic form of α of a similar type to that used in the truncated models. In addition to the presence of periodic behaviour in the system restricted to the invariant submanifold (top panel), these figures clearly show the presence of locking during the out phases (second panel) with an exponential growth of the energy of the transverse modes (bottom panel). This behaviour mirrors very closely the truncated model shown in Fig. 1 as well as that expected to occur from the theory [20].

3

Preliminary results show that the scaling for the PDE model is also compatible with (3), but a clear verification requires a much longer integration. We shall return to a more detailed study of this issue in the future. The presence of the main features necessary for the occurrence of in–out intermittency in these models, as well as its presence in related truncated models, constitutes strong evidence for the occurrence of in–out intermittency in these PDE models.
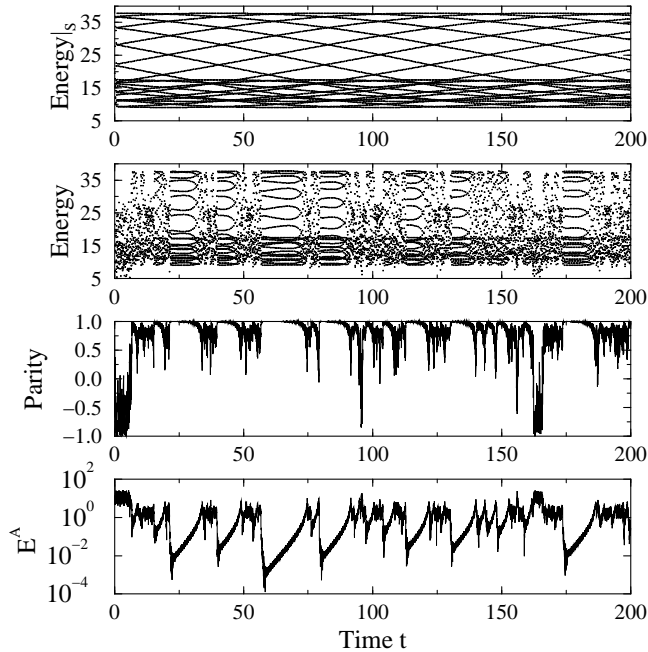

FIG. 4. In–out intermittency in the axisymmetric PDE mean–field dynamo model (1). The parameters used were $r_0 = 0.4$, $C_\alpha = 1.5$, $C_\Omega = -10^5$, $f = 0.7$, together with an algebraic form of α due to Kitchatinov [24]. The two upper panels are shown as in Fig. 3.

We note that similar types of behaviour have also been seen in mean–field dynamo models with different topologies [18,25] thus lending further support to the suggestion that such behaviour is likely to occur in other settings.

In summary, we have found strong evidence for the occurrence of in–out intermittency in both truncated and PDE axisymmetric mean–field dynamo models. Given the specific signatures of such intermittent regimes, this makes it in principle possible to test the presence of such behaviour in solar and stellar observations.

We also note that since dynamical systems are generically non–skew product with non–normal parameters, we expect this type of intermittency to be present in models of other physical systems with symmetry as well as their truncations.

[1] D. O. Gough, *Phil. Trans. R. Soc. Lond.* **A330**, 627 (1990).
[2] J. A. Eddy, *Science* **192**, 1189 (1976).
[3] S. L. Baliunas *et al.*, *Ap.J.* **438**, 269 (1995).
[4] R. Tavakol, *Nature* **276**, 802 (1978).
[5] B. Ya. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, in *Magnetic Fields in Astrophysics*, (Gordon and Breach, New York, 1983).
[6] E. A. Spiegel, in *Chaos in Astrophysics*, edited by J. R. Butcher, J. Perdang, and E. A. Spiegel (Reidel, Dordrecht, 1985).
[7] U. Feudel, W. Jansen, and J. Kurths, *Int. J. of Bifurcation and Chaos* **3**, 131 (1993).
[8] S. M. Tobias, N. O. Weiss, and V. Kirk, *MNRAS* **273**, 1150 (1995).
[9] N. O. Weiss, F. Cattaneo, and C. A. Jones, *Geophys. Astrophys. Fluid Dyn.* **30**, 305 (1984).
[10] A. A. Ruzmaikin, *Comm. Astrophys.* **9**, 88 (1981).
[11] N. O. Weiss, *Phil. Trans. R. Soc. Lond.* **A330**, 617 (1990).
[12] P. A. Gilman, *ApJ. Suppl.* **53**, 243 (1983).
[13] A. Brandenburg, I. Tuominen, F. Krause, R. Meinel, and D. Moss, *A&A* **213**, 411 (1989); A. Brandenburg, D. Moss, and I. Tuominen, *Geophys. Astrophys. Fluid Dyn.* **40**, 129 (1989).
[14] E. Covas and R. Tavakol, *Phys. Rev. E.* **55**, 6641 (1997).
[15] E. Covas, P. Ashwin, and R. Tavakol, *Phys. Rev. E.* **56**, 6451 (1997).
[16] R. Tavakol, A. S. Tworkowski, A. Brandenburg, D. Moss, and I. Tuominen, *A&A* **296**, 269 (1995).
[17] A. Tworkowski, R. Tavakol, A. Brandenburg, J. M. Brooke, D. Moss, and I. Tuominen, *MNRAS in press* (1998).
[18] J. M. Brooke, J. Pelt, R. Tavakol, and A. Tworkowski, *A&A* **332**, 339 (1998).
[19] E. Covas, R. Tavakol, A. Tworkowski, and A. Brandenburg, *A&A* **329**, 350 (1998).
[20] P. Ashwin, E. Covas, and R. Tavakol, submitted to *Nonlinearity* (1998). Preprint available at web address *http://www.maths.qmw.ac.uk/∼eoc*
[21] F. Krause and K.-H. Rädler, *Mean-Field Magnetohydrodynamics and Dynamo Theory*, (Pergamon, Oxford, 1980).
[22] P. Ashwin, J. Buescu, and I. Stewart, *Nonlinearity* **9**, 703 (1996); N. Platt, E. Spiegel, and C. Tresser, *Phys. Rev. Lett.* **70**, 279 (1993); E. Spiegel, N. Platt, and C. Tresser, *Geophys. Astrophys. Fluid Dyn.* **73**, 146 (1993).
[23] J. M. Brooke, *Europhysics Letters* **37**, 3 (1997).
[24] L. L. Kitchatinov, *Geophys. Astrophys. Fluid Dyn.* **38**, 273 (1987).
[25] J. M. Brooke and D. Moss, *A&A* **303**, 307 (1995).